Linearized Viscous Free Mixing with Streamwise Pressure Gradients

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Sets of similar solutions applicable within a linearized approximation are derived for two-dimensional compressible and axisymmetric incompressible viscous free mixing with streamwise pressure gradients Viewed as eigenfunctions these solutions can be superposed to represent arbitrary initial conditions However, it is observed that only one solution of the family possesses a net momentum defect (or excess) different from zero This mode decays exponentially in the normal direction and has a stronger streamwise persistence than any of the other solutions that have exponential normal decays; it is usually referred to as "the" similar solution

I Introduction

THIS paper concerns some pressure-gradient effects in viscous laminar wake-like or jet-like flows, an aspect of free-mixing problems that has received rather limited treatment in the literature One difficulty has stemmed from the fact that, in the configuration that has been studied, namely, the asymmetric shear layer between two semi-infinite two-dimensional streams,[1-6] similarity is exhibited only in a rather special case In this configuration the absence of similarity for the incompressible case has been demonstrated,[1] whereas similarity is achieved with the additional freedom provided by compressibility only if the Mach numbers in the two freestreams are identical at each streamwise station, evidently a strong restriction Nonsimilar incompressible flows produced by the two semi-infinite streams have been treated by series expansion procedures [2]

It is well-known that symmetric free-mixing configurations in uniform pressure wakes achieve similarity only asymptotically downstream, that is, as the velocity defect $(u_e - u)$ becomes small This was first demonstrated by Tollmein (Ref 7, p 138) who linearized the convective terms in the momentum boundary-layer equations, which are assumed to govern the phenomenon (The entire discussion in the present paper is confined to flows that can be described in terms of the boundary-layer theory) In fact, comparison with more exact series calculations such as those of Goldstein[8] and the integral method solution[9] show that the streamwise decay law of the maximum velocity defect, i e , the defect at the symmetric axis $(u_e - u_0)$, predicted by the linearized theory provides a reasonable approximation in the upstream regions where the defect may not be "so small " Clearly, if additional accuracy is required, the linearized solution can always be considered as a first approximation in a series expansion method wherein the expansion is in powers of the velocity defect Also, it may be recalled that the linearized equation, which is parabolic in nature, is of the familiar unsteady heat-conduction type, separable and thus solvable by well-developed procedures for a variety of initial conditions (see Refs 10–12)

The present work represents an extension of the forementioned linearized boundary-layer approach to the case of streamwise pressure gradients in two-dimensional symmetric and axially symmetric flows

The initial profiles may be wake-like, with velocities less than those of the freestream, or jet-like, with velocities in excess of those in the freestream, as indicated schematically in Fig 1 However, it should be noted that a primary assumption in the analysis is that the velocity defect, i e , $u_e - u$, be small compared to the external velocity u ; therefore, the case of a pure jet in a motionless ambient atmosphere cannot be included (Exact solutions of the isobaric jet are given in Ref 13)

Compressibility is accounted for only in cases for which the familiar Crocco integral of the energy equation applies (which requires a Prandtl and Lewis number equal to unity) Within this approximation, the stagnation enthalpy must be uniform in pressure-gradient cases, although it may be linearly related to the velocity, and therefore not necessarily uniform in the isobaric case Although the energy integral is applicable in both two-dimensional and axially-symmetric cases, its use here is limited to the former Furthermore only the case of uniform stagnation enthalpy is utilized

After this work had been completed, a paper by Stewartson[14] concerning similar, symmetric, two-dimensional, incompressible, wake-like solutions came to the authors' attention He considered the Falkner-Skan equation (in the nomenclature of Ref 7, p 128) $f''' + ff'' + (2m/m + 1)(1 - f'^2) = 0$, where $u/u = f'$, $u \sim x^m$, and primes denote total differentiation, subject to the boundary conditions $f(0) = f''(0) = 0$ and $f'(\infty) = 1\ 0$ Nontrivial solutions were derived for $-0\ 5 < (2m/m + 1) < 0$ These solutions correspond to case (*e*) of item (14) in the following section

The analog of the Falkner-Skan equation for axisymmetric, wake-like flow also warrants consideration It is obtained by letting $u/u = g'/n$, where $u_e \sim x^\xi$ and n is defined by Eq (29) This yields $[n(g'/n)]' + g(g'/n)' + \xi[n - (g'^2/n)] = 0$, subject to the boundary conditions at $n = 0$, $g/n = (g'/n)' = 0$ and as $n \to \infty$, $g'/n = 1\ 0$ This is equivalent to the case of constant u_0/u_e in Eq (31)

Furthermore, an approximate treatment, analogous to a momentum integral method, concerning the effects of pres-

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sure gradient on an axisymmetric, laminar hypersonic wake has been considered by Lykoudis [15]

II Analysis

The following boundary-layer equations are assumed to govern the viscous free-mixing previously discussed and represented schematically in Fig 1:

Continuity

$$(\rho u y^j)_x + (\rho v y^j)_y = 0 \tag{1}$$

Momentum

$$\rho u u_x + \rho v u_y = y^{-j}(\mu y^j u_y)_y + \rho\, u\, u\,_x \tag{2a}$$

$$p_y = 0 \tag{2b}$$

Energy

$$H = h + (u^2/2) = H \;\; = \text{const} \tag{3}$$

State

$$\rho/\rho \;\; = h\;/h \tag{4}$$

where j is an index equal to zero for two-dimensional flow and one for axially symmetric flow, x and y are the axial and normal coordinates with corresponding velocity components u and v, H is the stagnation enthalpy, ρ the density, p the pressure, μ the absolute viscosity, subscripts x and y denote partial differentiation, and subscript e denotes inviscid conditions at the outer edge of the viscous region and may be a function of x

The appropriate boundary conditions are

$$\text{at } y = 0 \qquad u_y = v = 0 \tag{5a}$$

and

$$\text{as } y \to \delta \qquad u = u_e \tag{5b}$$

where δ denotes the edge of the viscous layer

A Two-Dimensional Flow ($j = 0$)

As in the treatment of hypersonic boundary layers,[16] the following transformation is introduced:

$$n = \frac{u_e}{(2s)^{1/2}} \int_0^y \rho dy \tag{6a}$$

$$s = \int_0^x \rho\; \mu_e u_e dx \tag{6b}$$

Equation (1) is satisfied by the introduction of a stream function ψ, namely,

$$\rho u = \psi_y = [\rho u\;/(2s)^{1/2}]\psi_{\,n} \tag{7a}$$

$$\rho v = -\psi_x = -\rho\;\mu\; u\;[\psi \;+\; \psi_n n\;] \tag{7b}$$

As a result of well-known operations, Eq (2a) is expressed as follows:

$$(2s)^{1/2}(\psi_n u \;\; - \psi\; u_n) = (Gu\;)_n + (\rho\;/\rho)(2s)u \tag{8}$$

where $G = \rho\mu/\rho\;\mu$

Let

$$\begin{aligned} u - u \;\; &= (-1)^k \bar{u} = (-1)^k \bar{u}_{0\alpha} f_\alpha'(n) \\ f_\alpha'(n) &= (\bar{u}/\bar{u}_{0\alpha}) \end{aligned} \tag{9a}$$

and, therefore, from (7a) it follows that

$$\psi = (2s)^{1/2}[n + (-1)^k(\bar{u}_{0\alpha}/u\;)f_\alpha(n)] \tag{9b}$$

where subscript α denotes variables associated with one particular solution, α being an eigenvalue defined by Eq (12) Primes denote total differentiation with respect to n, subscript 0 denotes the velocity defect ($\bar{u}$) at the axis,

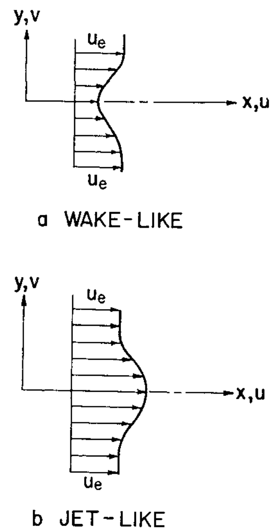

Fig 1 Schematic of axial section of free-mixing flow

$k = 1\;0$ for wake-like flows (wherein velocities are less than those of the freestream), and $k = 2$ for jet-like flows (i e , with velocities in excess of the freestream)

By utilizing (3, 4, and 9), Eq (8) reduces to the following:

$$\begin{aligned} [Gf_\alpha'']' + nf_\alpha'' + (\,\alpha - \beta)f_\alpha' &= -(-1)^k(\bar{u}_{0\alpha}/u\;)\times \\ &\{(1 - \alpha - \beta_i)f_\alpha f_\alpha'' + [\alpha - \beta_i(u\;^2/2h\;)]f_\alpha'^2\} \end{aligned} \tag{10}$$

where β is the pressure-gradient parameter; it is given by

$$\beta = \frac{2s}{u^!}\frac{du}{ds}\left[1 + \frac{u_e^2}{h_e}\right] \tag{11}$$

β_i is the value of β when $(u_e^2/h_e) \to 0$ and α is an eigenvalue, which, when properly evaluated, permits one to obtain a solution that has the properly bounded behavior as n approaches infinity and is defined by

$$\alpha = -(2s/\bar{u}_{0\alpha})(d\bar{u}_{0\alpha}/ds) \tag{12}$$

The appropriate boundary conditions are

$$f_\alpha(0) = f_\alpha''(0) = f_\alpha'(\infty) = 0 \tag{13}$$

For brevity, the subscript α will be omitted hereafter

The requirements for similarity are

$$G = G(n) \text{ or const} \tag{14a}$$

$$\alpha = \text{const} \tag{14b}$$

$$\beta = \text{const} \tag{14c}$$

$$(-1)^k \frac{\bar{u}_0}{u}\left[(1 - \alpha - \beta_i)ff'' + \left(\alpha - \beta_i \frac{u\;^2}{2h}\right)f'^2\right] \tag{14d}$$

is of the order of ϵ if $[(Gf'')' + nf'' + (\alpha - \beta)f']$ is of the order of unity, where ϵ is a constant whose magnitude is much less than unity Alternative to (14d),

$$u_0/u \;\; = \text{const} \tag{14e}$$

Here it is assumed that $\rho\mu = \rho\;\mu_e$ and, therefore, (14a) yields $G = 1\;0$ Conditions (14b) and (14c) give, respectively,

$$\bar{u}_0/\bar{u}_0 \;\; = [s\;/s]^{\alpha/2} \tag{15}$$

$$[u_e/u\;\;][h\;\;/h\;] = [s/s\;]^{\beta/2} \tag{16}$$

where subscript c denotes conditions at an initial station The dependence of $\bar{u}_0$ and u on the physical distance x will be discussed later

Condition (14d) can, in general, always be rendered valid by making $\bar{u}_0/u_e$ sufficiently small The implications of this condition will be examined at the end of this section Herein, only condition (14d) is studied

It is of interest to note that Eq (14d) indicates that to the first approximation the solution for wake-like and jet-like flows are identical However, if the second approximation to a series expansion is attempted, its solution will depend on the type of flow, since the nature of the flow is retained in the terms that are of order ϵ

By postulating conditions (14a–14d), Eq (10) becomes

$$f''' + nf'' + (\alpha - \beta)f' = 0 \tag{17}$$

Equation (17) with the boundary conditions (13) constitutes a Sturm-Lionville problem with eigenfunctions f_α and eigenvalues α

Equation (17) is of the confluent hypergeometric type[17] and has the following series solution:

$$f' = e^{-\xi}\left[1 - \frac{c}{\gamma}\,\xi + \frac{c(c-1)}{\gamma(\gamma+1)}\frac{\xi^2}{2!} - \frac{c(c-1)(c-2)}{\gamma(\gamma+1)(\gamma+2)}\frac{\xi^3}{3!} + \right] \tag{18}$$

where $\xi = n^2/2$, $c = (\alpha - \beta - 1)/2$, and $\gamma = \frac{1}{2}$ The second complementary solution to (17) is not regular at $n = 0$ and therefore can be discarded

It is readily seen that (18) satisfies the boundary conditions $f(0) = f''(0) = 0$ However, in order that $f'(\infty) = 0$ be satisfied, the eigenvalues α must be restricted The asymptotic behavior of f' is given in Ref 17 When c is a nonnegative integer $f' \to n^{2c}e^{-n^2/2}$ as $n \to \infty$, and therefore $f'(\infty) \to 0$ exponentially When c is a negative integer, $f'(\infty)$ is unbounded For nonintegral values of c, $f' \to n^{\beta-\alpha}$ as $n \to \infty$, and therefore $f'(\infty) \to 0$ algebraically when $\alpha > \beta$

Although f' is restricted, such that $\alpha > \beta$, there still exists an infinite continuous set of eigenvalues and associated eigenfunctions that satisfy the differential equation (17) and boundary conditions (13) Goldstein[18] suggests that particular solutions that decay algebraically at large n should be omitted However, there still would remain an infinite set of exponential solutions

Herein, primary interest is in the solution for which the linearized momentum and mass defect integral do not vanish identically, and is usually termed "the" similar solution ‡ The extraction of this solution can be achieved in a number of ways If it is observed that the momentum integral condition must be satisfied, namely,

$$\frac{d}{dx}\int_0^\infty \rho u(u - u)dy = -\int_0^\infty (\rho\, u_e - \rho u)dy\,\frac{du_e}{dx} \tag{19}$$

it can be shown that within the linearized approximation Eq (19) requires that

$$\alpha = 1 + \beta \tag{20}$$

unless

$$\int_0^\infty (u - u)dn = 0$$

‡ The general solution for arbitrary initial conditions and arbitrary pressure gradient may be given in the physical plane; its analysis is described briefly here The momentum equation linearized with respect to velocity perturbations but wherein density variations may still be large is $\rho(u\,\bar{u})_x = (\mu\bar{\mu}_y)_y - \rho[(u_e/h)\bar{u}]u_e u_{\,x}$, where $\bar{u} = u_e - u$, and from (3) and (4) $[(\rho/\rho) - 1] \cong (u\,\bar{u}/h)$ By letting

$$\eta = \int_0^y \rho dy \qquad z = \int_0^x \left(\frac{\rho_e\mu_e}{u}\right)dx$$

$$h_e = H_e - \frac{u_e^2}{2} \qquad \rho\mu = \rho\,\mu \qquad \varphi = \frac{u_e\bar{u}}{h}$$

the governing equation reduces to $\varphi = \varphi_{\eta\eta}$ which is linear and whose solution is readily given in terms of integral transforms

The condition imposed on the eigenfunction f_α by Eq (19) can also be expressed by means of direct integration of Eq (17) with $f_\alpha''(\infty) = 0$, as follows:

$$(\alpha - \beta - 1)f_\alpha(\infty) = 0 \tag{21}$$

This relation is satisfied when $\alpha = 1 + \beta$, although $f_{1+\beta}(\infty) \neq 0$ It is also satisfied when $\alpha \neq 1 + \beta$, since $f_\alpha(\infty) = 0$ in this case This may be interpreted by recalling that the momentum defect in the linearized approximation is proportional to a sum $A_\alpha(x)f_\alpha(\infty)$, which is thus seen to have only one nonzero contribution, that of $\alpha = 1 + \beta$ The additional eigenfunctions, when summed, provide foi the representation of arbitrary initial conditions, but their contribution to the total momentum defect is zero Moreover, the mode $\alpha = 1 + \beta$ decays most slowly if the admissible solutions are restricted to those that decay exponentially at $n \to \infty$

Therefore, with $\alpha = 1 + \beta$, Eq (9a) becomes

$$u - u_e = (-1)^k\bar{u}_{0_c}(s/s)^{1+\beta/2}\,e^{-n^2/2} \tag{22}$$

where u_e is given by Eq (16)

The relation between s and x is derived from Eqs (6b) and (16) Several features of the solution are presented below For brevity, the discusssion is limited to incompressible flows

In the particular case ρ = const, Eq (16) yields, for $\beta \neq 2$,

$$u = u\,(x/x_c)^{\beta/2-\beta} \tag{23a}$$

and for $\beta = 2$,

$$u_e = u\,e^{x/x} \tag{23b}$$

The solution for the streamwise velocity is given as follows:

$$\beta \neq 2 \qquad u - u = (-1)^k\bar{u}_{0_c}(x/x)^{1+\beta/2-\beta}e^{-n^2/2} \tag{24a}$$

and

$$\beta = 2 \qquad u - u_e = (-1)^k\bar{u}_{0_c}e^{-3x/2x}\,e^{-n^2/2} \tag{24b}$$

where u is given by Eqs (23)

In order that condition (14d) be satisfied for all x, it is required that $\bar{u}_{0_c}$ be small compared to u and $-1 \leq \beta \leq 2$ However, for limited ranges of x, other values of β are admissible (noting that when $\beta < -1$ or $\beta > 2$ and $x \gg x_c$ the velocity defect is no longer small) The permissible magnitude of $\bar{u}_0$ is discussed below

The region in which the solution is valid is now briefly discussed Equation (10) is recast as follows:

$$f''' + [n - 2(-1)^k(\bar{u}_0/u_e)\beta_i f]f'' + [1 + (-1)^k(\bar{u}_0/u)\,(1+\beta_i)f']f' = 0 \tag{25}$$

where use has been made of Eq (20)

Therefore, the linearized approximation requires

$$n \gg \frac{\bar{u}_0}{u}\,2\beta_i\int_0^n e^{-n^2/2}dn \tag{26a}$$

and

$$1 \gg (\bar{u}_0/u)(1 + \beta_i)e^{-n^2/4} \tag{26b}$$

In uniform-pressure flows Eq (26a) is always satisfied, whereas (26b) reduces to

$$1 \gg (\bar{u}_0/u)e^{-n^2/2} \tag{27}$$

This inequality basically describes the nonsimilar region and, moreover, shows that the error is a maximum at $n = 0$ (implying $\bar{u}_0 \ll u$) and diffuses as $(\bar{u}_0/u_e)e^{-n^2/2}$

In pressure-gradient flows the maximum inequality also is at the axis Equation (26a) gives

$$1 \gg 2(\bar{u}_0/u)\,(\text{absolute value of } \beta_i) \tag{28a}$$

whereas (26b) yields

$$1 \gg (\bar{u}_0/u\,)(1 + \beta_i) \tag{28b}$$

B Axisymmetric Flow ($j = 1\ 0$)

Axisymmetric flows can be treated in an analogous manner However, in the axisymmetric case the analysis must be restricted to incompressible flow, since there does not exist an appropriate compressibility transformation

Let

$$n = y(u_e/x\nu)^{1/2} \qquad s = x \tag{29}$$

and

$$u - u_e = (-1)^k\bar{u} = (-1)^k\bar{u}_{0a}\,\frac{F'(n)}{n} \qquad \frac{\bar{u}}{\bar{u}_{0a}} = \frac{F'(n)}{n} \tag{30}$$

where subscript (a) denotes quantities associated with axisymmetric flows

By using (29) and (30), Eqs (1) and (2) reduce to the following:

$$\left[n\left(\frac{F'}{n}\right)'\right]' + \frac{n^2}{2}\left[\frac{F'}{n}\right]' + (\alpha_a - \beta_{i_a})F' = \\ -(-1)^k\,\frac{\bar{u}_{0a}}{u_e}\left[(1 - \alpha_a - \beta_{i_a})F\left(\frac{F'}{n}\right)' + \alpha_a\,\frac{F'^2}{n}\right] \tag{31}$$

where

$$\alpha_a = -\frac{s}{\bar{u}_{0a}}\frac{d\bar{u}_{0a}}{ds} \qquad \beta_{i_a} = \frac{s}{u_e}\frac{du_e}{ds} \tag{32}$$

Conditions similar to Eqs (14b–14d) require that

$$\left[n\left(\frac{F'}{n}\right)'\right] + \frac{n^2}{2}\left[\frac{F'}{n}\right]' + (\alpha_a - \beta_{i_a})F' = 0 \tag{33}$$

and

$$\bar{u}_{0a}/\bar{u}_{0a} = [x\ /x]^{\alpha_a} \tag{34a}$$

$$u_e/u_{\ c} = [x/x_c]^{\beta_{i_a}} \tag{34b}$$

The appropriate boundary conditions are at $n = 0$, $F = 0$, $(F'/n)' = 0$; as $n \to \infty$, $(F'/n) = 0$

The solution to (33) is

$$F' = ne^{-n^2/4}\mathfrak{F}(1 + \beta_{i_a} - \alpha_a, 1; n^2/4) \tag{35}$$

where $\mathfrak{F}(1 + \beta_{i_a} - \alpha_a, 1; n^2/4)$ is the confluent hypergeometric function The boundary conditions are satisfied when $\alpha_a > \beta_{i_a}$

The similar solution§ may be extracted by direct integration of (33) The integration yields, with $(F'/n)' = 0$ as $n \to \infty$, the following:

$$(\alpha_a - \beta_{i_a} - 1)F(\infty) = 0 \tag{36}$$

Within the linearized approximation the momentum thickness is proportional to the sum $B_{\alpha_a}(x)F_{\alpha_a}(\infty)$, which by (36) is seen to have only one nonzero contribution, that of

$$\alpha_\alpha = 1 + \beta_{i_a} \tag{37}$$

The eigenfunction associated with (37) is termed the similar solution and is given by

$$F' = ne^{-n^2/4} \tag{38}$$

Equation (30) then becomes

$$u - u\ = (-1)^k\bar{u}_0\,(x_c/x)^{1+\beta_{i_a}}\,e^{-n^2/4} \tag{39}$$

where u_e is given by Eq (34b)

If the solution is restricted to flows wherein $\bar{u}_0$ is small everywhere compared to u , then it is required that $\beta_{i_a} \geq -1$ and $\bar{u}_0 \ll u_e$

§ In Ref 19 (Sec E 1) the incompressible linearized solution is given in the physical plane for arbitrary initial conditions and for both axisymmetric and three-dimensional wakes

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